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A Lower Estimation of the Hausdorff Dimension for Attractors with Overlaps

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We give a lower estimate of the Hausdorff dimension for attractors which can be obtained by an overlapping construction.

KEY WORDS: Hausdorff dimension; fractal; iterated function system.

1. INTRODUCTION AND NOTATION

Let (X, ρ) be a Polish space, i.e., a separable complete metric space. By $B(x, r)$ we denote the closed ball in X with center at x and radius r. For $A \subset X$, $A \neq \emptyset$, we denote by diam A the diameter of A. As usual, $\mathbb{R}$ stands for the set of all reals and $\mathbb{N}$ for the set of all positive integers. Moreover set $\mathbb{R}_+ = [0, +\infty)$.

For $A \subset X$, $s > 0$ and $\delta > 0$ we define

$$\mathscr{H}^s_\delta(A) = \inf \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s,$$

where the infimum is taken over all countable covers $\{U_i\}$ of A such that $\operatorname{diam} U_i \leqslant \delta$. Then

$$\mathscr{H}^s(A) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(A)$$

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is *Hausdorff s-dimensional measure*. The *Hausdorff dimension* of A is defined by the formula

$$\dim_H A = \inf\{s > 0 \colon \mathscr{H}^s(A) < \infty\}.$$

Unfortunately, the Hausdorff dimension of even relatively simple sets is rather hard to calculate.

Given a set $\{S_1,..., S_N\}$ of strictly contractive mappings $S_i \colon X \to X$, we define a mapping F on the subsets of X, by the formula

$$F(A) = \bigcup_{i=1}^{N} S_i(A) \qquad \text{for} \quad A \subset X.$$

It is well known (see [H]) that there exists a unique non-empty compact set K such that $F(K) = K$ and for every nonempty compact subset A of X the sequence $\{F^n(A)\}$ converges in the Hausdorff metric to the set K. The set K is called *attractor* or *fractal* of the iterated function system $\{S_1,..., S_N\}$. From the point of applications it is convenient to have estimates of the Hausdorff dimension of the set K. Upper bounds can easily be obtained, while lower bounds are much more difficult to establish. Here we obtain an estimate using the mass distribution principle formulated by Frostman in 1935 (see [Fr]) and a version of Lasota's lemma.

We say that the system $\{S_1,..., S_N\}$ satisfies the *Moran condition* if the sets $S_1(K),..., S_N(K)$ are pairwise disjoint, where K is the attractor of the system. If the system $\{S_1,..., S_N\}$ satisfies the Moran condition and

$$\rho(S_i(x), S_i(y)) \geqslant l_i \rho(x, y) \qquad \text{for} \quad x, y \in X \quad \text{and} \quad i = 1,..., N, \tag{1}$$

then the Hausdorff dimension of the attractor K of this system (see [M, H]) is greater than or equal to the unique number d given by

$$l_1^d + \cdots + l_N^d = 1. \tag{2}$$

Moreover, if $S_1,..., S_N$ are similarities with the scaling factors $l_1,..., l_N$, respectively, then the Hausdorff dimension of K is equal to d. If $X = \mathbb{R}^d$, this remains true if the sets $S_i(K)$, $i = 1,..., N$, have "small overlap". To define this we use so called open set condition. We say that the system $\{S_1,..., S_N\}$ satisfies the *open set condition* if there exists a nonempty open set G such that the sets $S_1(G),..., S_N(G)$ are pairwise disjoint and $S_i(G) \subset G$ for $i = 1,..., N$. Since the intersection of G and K may be empty, the open set condition does not imply the Moran condition. There are some results (see [F], [PS], [Si]) which show that for a family of similarities with overlaps, the Hausdorff dimension is almost surely equal to d. However, the case with overlaps has not been completely analysing. In this Note we give a contribution to this.

2. RESULTS

First we recall the mass distribution principle (see [Fr, O, P]).

Proposition 1. Let Z be a subset of X and μ a Borel finite measure on Z. Assume that there exist numbers $s>0$ and $C>0$ such that for μ-almost all $x \in Z$

$$\mu(B(x,r)) \leqslant Cr^s \qquad \text{for} \quad r>0.$$

Then

$$\dim_H Z \geqslant s.$$

An essential tool in our proof is the following lemma concerning functional inequalities, based on an idea of A. Lasota (see [LM]).

Lemma 2. Let $l_i, p_i \in (0,1)$ for $i=1,\ldots,N$ be given. Assume that $\sum_{i=1}^N p_i = 1$. Let Λ be a family of sequences of integers $(i_1,\ldots,i_m)$, $m<N$, such that $1 \leqslant i_1 < \cdots < i_m \leqslant N$. Let $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded function. Suppose that there exists $r_0>0$ such that

$$\Phi(r) \leqslant \max_{(i_1,\ldots,i_m)\in\Lambda} \sum_{k=1}^m p_{i_k} \Phi(r/l_{i_k}) \qquad \text{for} \quad r \in (0, r_0]. \tag{3}$$

Then for $s>0$ satisfying

$$\max_{(i_1,\ldots,i_m)\in\Lambda} \sum_{k=1}^m \frac{p_{i_k}}{l_{i_k}^s} = 1, \tag{4}$$

there exists $C>0$ such that

$$\Phi(r) \leqslant Cr^s \tag{5}$$

for every $r>0$.

Proof. Let Λ, Φ and r_0 be as in Lemma 2. Fix $s>0$ such that (4) holds. Since Φ is bounded, there exists $C>0$ such that (5) holds for every

$r \geqslant r_0$. Let $i_0 \in \{1,..., N\}$ be such that $l_{i_0} = \max_{1 \leqslant i \leqslant N} l_i$. We claim that (5) holds for every $r \geqslant l_{i_0}^n r_0$, where n is an arbitrary nonnegative integer. To prove this we will use induction. For $n = 0$ inequality (5) is true for every $r \geqslant r_0$ by the choice of C. Suppose now that (5) holds for $r \geqslant l_{i_0}^n r_0$. From this and the inequality $r/l_i \geqslant l_{i_0}^n r_0$ for $r \geqslant l_{i_0}^{n+1} r_0$, $i = 1,..., N$, we have

$$\Phi(r) \leqslant \max_{(i_1,..., i_m) \in \Lambda} \sum_{k=1}^{m} p_{i_k} \Phi(r/l_{i_k}) \leqslant \max_{(i_1,..., i_m) \in \Lambda} \sum_{k=1}^{m} p_{i_k} C r^s / l_{i_k}^s$$

$$= C r^s \max_{(i_1,..., i_m) \in \Lambda} \sum_{k=1}^{m} p_{i_k} / l_{i_k}^s.$$

Consequently, from (4) it follows that (5) holds for every $r \geqslant l_{i_0}^{n+1} r_0$. Thus (5) holds for every $r \geqslant l_{i_0}^n r_0$ and every $n \in \mathbb{N}$. Since $l_{i_0} < 1$ it follows that (5) holds for every $r > 0$. The proof is complete. ▮

We are now in a position to formulate the main theorem:

Theorem 3. Let $S_1,..., S_N$ be strictly contractive mappings satisfying (1) with $l_i \in (0, 1)$, $i = 1,..., N$. Let K be the attractor of the system $\{S_1,..., S_N\}$. Assume that

$$\bigcap_{i=1}^{N} S_i(K) = \emptyset. \tag{6}$$

Let Λ be the family of all sequences of integers $(i_1,..., i_m)$ such that $1 \leqslant i_1 < \cdots < i_m \leqslant N$ and

$$\bigcap_{k=1}^{m} S_{i_k}(K) \neq \emptyset.$$

Then

$$\dim_H K \geqslant s,$$

where $s > 0$ satisfies

$$\max_{(i_1,..., i_m) \in \Lambda} \sum_{k=1}^{m} l_{i_k}^{d-s} = 1$$

for d given by (2).

Proof. For $x \in K$ define

$$I_x = \{i \in \{1,\dots, N\} : x \notin S_i(K)\}.$$

From (6) it follows that the set I_x is nonempty. Consider the function $\varphi: K \to \mathbb{R}_+$ given by

$$\varphi(x) = \max_{y \in K} \min_{i \in I_y} \rho(x, S_i(K)).$$

It is easy to verify that φ is continuous and positive. Thus

$$r_0 = \inf_{x \in K} \varphi(x) > 0.$$

Consider the probabilistic iterated function system $(S_1,\dots, S_N; p_1,\dots, p_N)$ with $p_i = l_i^d$, $i = 1,\dots, N$, where d is given by (2). It is well known (see [LM, Sz]) that there exists a Borel probability measure μ such that the support of μ is equal to K and

$$\mu(A) = \sum_{i=1}^{N} l_i^d \mu(S_i^{-1}(A)) \tag{7}$$

for every Borel subset A of X. Simple calculation shows that $S_i^{-1}(B(x, r)) \subset B(S_i^{-1}(x), r/l_i)$ for $i = 1,\dots, N$. Further there exists $y \in K$ such that $S_i^{-1}(B(x, r)) \cap K = \emptyset$ for $i \in I_y$ and $r \leqslant r_0$. From this and (7) it follows that

$$\mu(B(x, r)) \leqslant \sum_{i \in \{1,\dots, N\} \setminus I_y} l_i^d \, \mu(B(S_i^{-1}(x), r/l_i)) \qquad \text{for} \quad r \in (0, r_0]. \tag{8}$$

Consider the function $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\Phi(r) = \sup_{x \in K} \mu(B(x, r)).$$

From (8) it follows that

$$\Phi(r) \leqslant \max_{(i_1,\dots, i_m) \in A} \sum_{k=1}^{m} p_{i_k} \Phi(r/l_{i_k}) \quad \text{for} \quad r \leqslant r_0.$$

By Lemma 2 there is $C > 0$ such that

$$\Phi(r) \leqslant Cr^s \qquad \text{for} \quad r > 0,$$

where $s > 0$ satisfies condition (4) with $p_i = l_i^d$ for $i = 1,\dots, N$. Theorem 3 now follows from Proposition 1. ∎

Remark. It is easy to see that if we replace in (6) the attractor K by a set K_0 such that $\bigcup_{i=1}^{N} S_i(K_0) \subset K_0$, Theorem 3 remains true. Moreover, observe that condition (6) is not restrictive. In fact, if (6) is not satisfied, the system may admit only a one point attractor and clearly $\dim_H K = 0$.

Using Theorem 3 we can obtain the following:

Corollary 4. Let $S_1, \dots, S_N$ be strictly contractive mappings satisfying (1) with $l_i \in (0, 1)$, $i = 1, \dots, N$. Let K be the attractor of the system $\{S_1, \dots, S_N\}$. If $\{S_1, \dots, S_N\}$ satisfies the Moran condition, then

$$\dim_H K \geqslant d,$$

where d is given by (2).

Proof. It is enough to note that $\Lambda = \{\{1\}, \dots, \{N\}\}$. ▮

The following example shows that our theorem can be used in the case when the obvious approach of selecting a subset of the S_i for which $S_i(K)$ are pairwise disjoint and applying the usual formula due to Moran (see [H], [M], [Mo]) breaks down.

Example. Consider the affine transformations $S_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by the condition

$$S_i\begin{pmatrix} x \\ y \end{pmatrix} = A_i\begin{pmatrix} x \\ y \end{pmatrix} + a_i \qquad \text{for} \quad i = 1, 2, 3,$$

where

$$A_1 = \begin{pmatrix} 0.35 & 0.35 \\ -0.35 & 0.35 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.05 & 0.25 \\ 0.55 & 0.45 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0.30 & 0.455 \\ 0.45 & 0.795 \end{pmatrix},$$

$$a_1 = \begin{pmatrix} 0.15 \\ 0.50 \end{pmatrix}, \qquad a_2 = a_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Set $x_0 = \binom{0}{0}$ and $x_1 = \binom{0.5}{0.5}$. (See Fig. 1.)

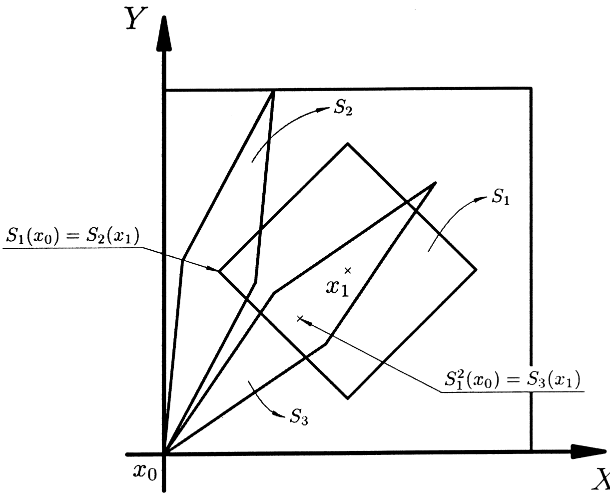

Fig. 1.

From the definition of S_i, $i = 1, 2, 3$, it follows that

$$S_1(x_0) = S_2(x_1), \quad S_2(x_0) = S_3(x_0), \quad \text{and} \quad S_3(x_1) = S_1^2(x_0).$$

Let K be the attractor of the system $\{S_1, S_2, S_3\}$. Obviously condition (6) is satisfied and $\{x_0, x_1, S_1(x_0)\} \subset K$. Thus we have

$$S_i(K) \cap S_j(K) \neq \varnothing \qquad \text{for} \quad i, j \in \{1, 2, 3\}.$$

Fix $i, j \in \{1, 2, 3\}$ and denote by K_{ij} the attractor of the system $\{S_i, S_j\}$. An argument similar to the above shows that

$$S_i(K_{ij}) \cap S_j(K_{ij}) \neq \varnothing.$$

Define

$$l_i = \inf\{\|A_i\mathbf{x}\|: \|\mathbf{x}\| = 1\}, \qquad \text{for} \quad i = 1, 2, 3$$

and observe that (1) holds. Then we have

$$l_i^2 = \inf\{\|A_i\mathbf{x}\|^2: \|\mathbf{x}\| = 1\} = \inf\{\langle A_i^* A_i \mathbf{x}, \mathbf{x}\rangle: \|\mathbf{x}\| = 1\}.$$

(Here $\langle \cdot, \cdot \rangle$ denotes the scalar product and A^* stands for the adjoint operator of A.) Hence we obtain (see [M1])

$$l_i^2 = \min\{\lambda_{i,1}, \lambda_{i,2}\},$$

where $\lambda_{i,1}$, $\lambda_{i,2}$ denote the eigenvalues of A_i.

Theorem 3 now gives

$$\dim_H K \geqslant s,$$

where $s > 0$ satisfies

$$\max_{i \neq j} l_i^{d-s} + l_j^{d-s} = 1$$

and d is given by (2).

The following numerical results were obtained with *Mathematica 3.0*. Namely, in our case we have:

$$l_1 \approx 0.495984, \qquad l_2 \approx 0.155660, \qquad l_3 \approx 0.031742$$

and

$$d \approx 0.1695985, \qquad s \approx 0.0234472.$$

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